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Asymptotic expansion of homoclinic structures in a symplectic mapping

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Abstract. The phenomenon of separatrix splitting and complicated homoclinic structures constitute a symptom of chaos, as pointed out by Poincaré more than a century ago. Taking a time-discrete dynamical system with a double-well potential as an example, this interesting feature is demonstrated analytically on the basis of *asymptotic expansions beyond all orders*. Violent undulations of the unstable manifold in the extreme vicinity of hyperbolic fixed points are recovered excellently. Comparison with results for standard and Henon maps is also made.

1. Introduction

The bifurcation of separatrices and homoclinic or heteroclinic structures are well known to lead to genesis of chaos in conservative dynamical systems [1]. While numerical iterations of low-dimensional mappings easily provide these complicated structures, it is extremely difficult to derive them analytically. However, the difficulty will now be overcome by using the *asymptotic expansion beyond all orders*. This method was first proposed and applied to a standard map by Lazutkin and coworkers [2, 3] and to other systems [4]. A similar approach was developed independently by Kruskal and Segur [5] in the context of crystal growth. The method was improved by using the theoretical tools of Borel summability and Stokes phenomenon [6–8]. However, this updated method is still in its infancy and its application can be found only in a few references, i.e. on standard [6] and Henon [8] maps. So it deserves to be further enriched by providing additional applications. In this paper we shall apply the updated method to the time-discrete dynamical system with a double-well potential.

We shall analyse a symplectic mapping obtained by time discretization of canonical equations for the dynamical system with a single degree of freedom. Consider the canonical equation of motion

$$\begin{aligned} \mathrm{d}q(t)/\mathrm{d}t &= p(t) \\ \mathrm{d}p/\mathrm{d}t &= -\mathrm{d}U(q)/\mathrm{d}q. \end{aligned} \tag{1}$$

For concreteness, we choose a double-well potential

$$U(q) = (q-a)^2(q+a)^2 \tag{2}$$

which is often encountered in describing the phase-transition phenomena. A set of equations (1) with equation (2) are widely seen in various contexts. The Duffing equation,

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in the absence of damping and driving fields, takes the identical form; it would also be reached if we substitute $\Psi(x,t) = e^{i\omega t} q(x)$ in the cubic nonlinear Schrödinger equation and then replace x by t in the resultant form. While the above ordinary differential equation is integrable, we shall consider its time-difference variant by discretization of time (with *time difference* σ). Among many ways of time discretization the symplectic mapping is the most essential, which is given by

$$\begin{aligned} q_{n+1} &= q_n + \sigma p_{n+1} \\ p_{n+1} &= p_n - \sigma\, \mathrm{d}U(q)/\mathrm{d}q|_{q=q_n} = p_n - 4\sigma q_n(q_n + a)(q_n - a). \end{aligned} \tag{3}$$

(By rescaling $\sigma p \to p$, (3) is reduced to the map available from a periodically kicked system where σ^2 plays the role of the kicking strength. In this paper, however, we shall choose the form (3) for convenience.)

The Jacobi matrix corresponding to (3) is

$$M = \begin{pmatrix} \frac{\partial q_{n+1}}{\partial q_n} & \frac{\partial q_{n+1}}{\partial p_n} \\ \frac{\partial p_{n+1}}{\partial q_n} & \frac{\partial p_{n+1}}{\partial p_n} \end{pmatrix} = \begin{pmatrix} 1 - 4\sigma^2(3q_n^2 - a^2) & \sigma \\ -4\sigma(3q_n^2 - a^2) & 1 \end{pmatrix}. \tag{4}$$

The present map is area-preserving since det $M = 1$, and satisfies the symplecticity condition $M^T J M = J$ with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The map (3) and its continuum (1) and (2) have the common fixed points, i.e. one hyperbolic at $(q,p) = (0,0)$ and the other two elliptic at $(q,p) = (\pm a, 0)$.

For $\sigma = 0$ (i.e. in the continuum limit) the phase space is occupied by regular trajectories including a separatrix, i.e. the marginal trajectory by which localized and extended trajectories are segregated (see figure 1(*a*)). The separatrix is the most unstable against a perturbation and consists of a pair of degenerate manifolds: one is unstable and going away from the hyperbolic fixed point (HFP) while the other is stable and coming into HFP. On switching a perturbation arising from $\sigma \neq 0$, the splitting of the separatrix occurs yielding infinite number of crossing points (: homoclinic points) that accumulate as HFP is approached (see figure 1(*b*)). The separatrix splitting and homoclinic structures cause Birkhoff–Smale's horse-shoe mechanism generating the chaos. We shall make the

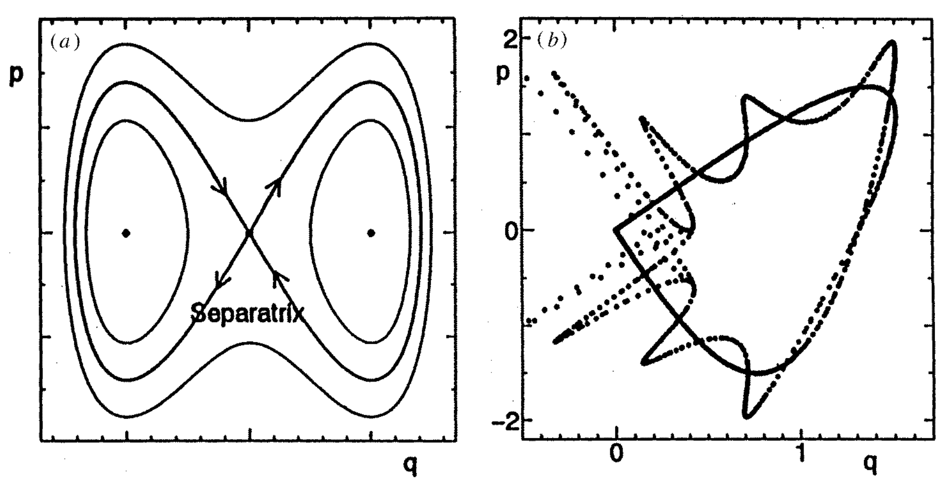

Figure 1. Trajectories in phase space. (*a*) $\sigma = 0$, separatrices are emerging from hyperbolic fixed point at (0, 0). (*b*) $\sigma \neq 0$, splitting of degenerate separatrices for the right half.

asymptotic analytical expansion of the unstable manifold emanating from HFP at $(0, 0)$. While we shall concentrate on the system with a double-well potential, the following analysis will hold to more general systems.

2. External equation

Prescribing q_n and q_{n+1} as $y(t)$ and $y(t+\sigma)$, respectively, let us concentrate on the unstable manifold $y_u(t)$ with the initial condition $\lim_{t\to-\infty} y_u(t) = 0$. For brevity, $y_u(t)$ will be taken as $y(t)$ in the following. A pair of equations (3) is then reduced to the second-order difference equation for $y(t)$:

$$\Delta^2 y(t)/\sigma^2 = -4y(t)(y(t) - a)(y(t) + a) \tag{5}$$

with $\Delta^2 y(t) = y(t - \sigma) + y(t + \sigma) - 2y(t)$.

Let us first attempt to apply the ordinary perturbation theory as proposed by Melnikov. Using Taylor expansion as

$$\Delta^2 y(t) = \sigma^2\, \mathrm{d}^2 y(t)/\mathrm{d}t^2 + 2\sum_{l=2}^{\infty}(\sigma^{2l}/(2l)!)y^{(2l)}(t)$$

one may rewrite (5) as

$$\mathrm{d}^2 y(t)/\mathrm{d}t^2 = -4y(t)(y(t) - a)(y(t) + a) - 2\sum_{l=2}^{\infty}\sigma^{2(l-1)}((2l)!)^{-1}y^{(2l)}(t). \tag{6}$$

Taking the last term on the r.h.s. of (6) as perturbations, we shall expand the solution $y(t)$ in the power series of σ^2 as

$$y(t) = y_0(t) = y_{00}(t) + \sigma^2 y_{01}(t) + \sigma^4 y_{02}(t) + \cdots. \tag{7}$$

The lower index of $y_0(t)$ is used to indicate that the solution in (7) will turn out to be incomplete without 'terms beyond all orders'. Using (7) in (6), equations are successively obtained for each power of σ^2:

$$\mathrm{d}^2 y_{00}(t)/\mathrm{d}t^2 = -4y_{00}(t)(y_{00}(t) + a)(y_{00}(t) - a) \tag{8a}$$

$$\mathrm{d}^2 y_{01}(t)/\mathrm{d}t^2 = -4(3y_{00}(t)^2 - a^2)y_{01}(t) - (1/12)\,\mathrm{d}^4 y_{00}(t)/\mathrm{d}t^4. \tag{8b}$$

For the unperturbed system (8*a*), we have

$$y_{00}(t) = \sqrt{2}a/\cosh(2at). \tag{9}$$

$y_{0n}(t)$ for $n = 1, 2, \ldots$ should bear the following properties:

(i) They should satisfy the boundary condition ensuring the orbit (constructed from (7)) to start from HFP at $(0, 0)$:

$$\lim_{t\to-\infty} y_{0n}(t) = 0 \qquad (n = 0, 1, \ldots).$$

(ii) Their parity is even: $y_{0n}(-t) = y_{0n}(t)$ for $n = 1, 2, \ldots$ because of the even-parity nature of the inhomogeneous term on the r.h.s. of (8*b*) and the uniqueness of the solution for $y_{0n}(t)$ for $n = 1, 2, \ldots$ under the unique initial condition.

The solution of (8*b*) is a sum of a general solution for its homogeneous part and the special one for the full inhomogeneous equation. The former is a linear combination of two independent solutions $v_1(t)$ and $v_2(t)$ given by

$$v_1(t) = -\frac{2\sqrt{2}a^2\sinh[2at]}{\cosh^2[2at]} \tag{10a}$$

$$v_2(t) = -\frac{1}{8\sqrt{2}a^3}\left(\frac{6at\sinh[2at]}{\cosh^2[2at]} - \frac{3}{\cosh[2at]} + \cosh[2at]\right). \tag{10b}$$

Any linear combination of (10*a*) and (10*b*), however, cannot satisfy both (i) and (ii) at the same time: $v_1(t)$ has an odd parity and is incompatible with (ii); $v_2(t)$ diverges with $t \to -\infty$. Hence the solution of (8*b*) is inevitably provided by the latter special contribution alone, i.e.

$$y_{01}(t) = \frac{a^3\sqrt{2}}{12\cosh^3[2at]}(9 - 7\cosh[4at] + 2at\sinh[4at]). \tag{11}$$

Because of the even-parity nature, the result (11) can lead to no splitting of the separatrix, which is not consistent with the issue of numerical iteration of the map (3) with $\sigma > 0$. This indicates a breakdown of the application of Melnikov's method.

The above paradox is caused by the singularities of (9) and (11) at $t = \pm\frac{\mathrm{i}\pi}{4a}(2m+1)$ with $m = 0, 1, \ldots$ encountered in changing the time t continuously from $t = -\infty$ to $t = +\infty$ in the complex time plane. In fact, in the neighbourhood of $t_c = \frac{\mathrm{i}\pi}{4a}$

$$y_{00} \sim \frac{c_0\mathrm{i}}{(t - t_c)} \qquad \text{with } c_0 = -\frac{\sqrt{2}}{2}. \tag{12a}$$

Similarly, reflecting the degree of the singularity of the inhomogeneous part of (8*b*), we find

$$\begin{aligned} &y_{01} \sim \frac{c_1\mathrm{i}}{(t - t_c)^3} \qquad \text{with } c_1 = \frac{\sqrt{2}}{6} \\ &\vdots \\ &y_{0n} \sim \frac{c_n\mathrm{i}}{(t - t_c)^{2n+1}} \qquad \text{with } c_n \sim (-1)^{n+1}(2n+3)!/(2\pi)^{2n+1} \qquad \text{for } n \to \infty. \end{aligned} \tag{12b}$$

The behaviour in (12) indicates that all orders in the expansion (7) give the contributions of the identical magnitude of $O(\sigma^{-1})$ at $|t - t_c| \sim \sigma$ and that the perturbation theory breaks down there. The crucial point is that we meet the Stokes phenomenon: (i) Stokes line is emanating from $t = t_c$; (ii) a suitable odd-parity correction should be incorporated in crossing this line.

To analyse this phenomenon and to capture the 'terms beyond all orders', we shall derive the *internal equation*, effective in the vicinity of $t = t_c$.

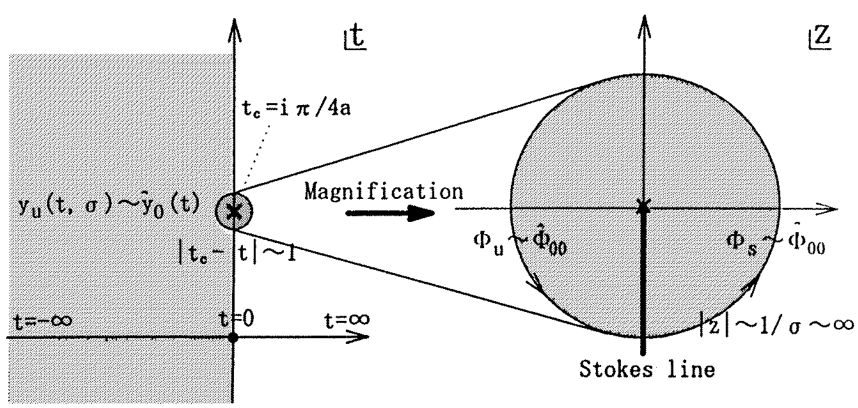

Figure 2. Transformation of vicinity of $t = t_c$ from the t plane to the enlarged z plane.

3. Internal equation

Let us enlarge a scale of time in the neighbourhood of $t = t_c$ and decrease the magnitude of $y(t)$ by making a transformation (see, figure 2) as

$$t = \frac{\mathrm{i}\pi}{4a} + \sigma z \tag{13a}$$

$$\Phi(z) = (\sigma/\mathrm{i})y(t). \tag{13b}$$

Using (13) in the original equation (5), we obtain the *internal equation*:

$$\Delta^2\Phi(z) = 4(\Phi^3(z) + \sigma^2 a^2 \Phi(z)) \tag{14}$$

with $\Delta^2\Phi(z) = \Phi(z+1) + \Phi(z-1) - 2\Phi(z)$. (Note: The tiny circle $|t - t_c| = 1$ is now mapped to the big one $|z| = 1/\sigma (\gg 1)$.) By suppressing the small contribution of $O(\sigma^2)$ which gives pre-exponential corrections in the result, (14) becomes a σ-independent equation

$$\Delta^2\Phi(z) = 4\Phi^3(z). \tag{15}$$

Let Φ_u and Φ_s be the solutions for unstable and stable manifolds respectively. In the limit $z \to \infty$ with $\mathrm{Re}\, z < 0$ ($\mathrm{Re}\, z > 0$), the solution of (15) has an asymptotic form

$$\Phi_u(\Phi_s) \sim \Phi_{00} = \sum_{l=0}^{\infty} \frac{c_l}{z^{2l+1}} \tag{16}$$

which retains the connection with the external solution (in the region with $z \to \infty$, $\sigma \to 0$ and $\sigma z \to 0$) (see figure 2):

$$(\sigma/\mathrm{i})y_u(t) \sim (\sigma/\mathrm{i})y_0(t) \sim \sum_{l=0}^{\infty} \frac{c_l \sigma^{2l+1}}{\left(t - \frac{\mathrm{i}\pi}{4a}\right)^{2l+1}} = \Phi_{00}(z). \tag{17}$$

For any finite value of z, however, c_l grows much larger than z^{2l+1} with increasing l (: $c_l \sim (-1)^{l+1}(2l+3)!/(2\pi)^{2l+1}$, as seen in (12$b$)) so that, except for $|z| = \infty$, (16) diverges and becomes meaningless. Since the asymptotic expansions at $z \to -\infty$ and $z \to \infty$ cannot, therefore, be connected smoothly so long as the finite z region is crossed, we shall take a counter-clockwise path along the lower semicircle with $|z| = \infty$. In this case the Stokes phenomenon appears: in crossing the Stokes line at $\arg(z) = -\pi/2$, we acquire an exponentially small term responsible for the separatrix splitting that is being searched for.

In this context we shall recourse to the idea of *Borel summation*. The Borel summation provides the way to find a convergent sum from divergent series by resorting to the Laplace transformation. (The idea is based on the resummation of divergent series by a suitable reordering of the terms.) The Borel or inverse Laplace transformation of (15) yields

$$2(\cosh[p] - 1)V(p) = 4V(p) * V(p) * V(p) \tag{18}$$

where we have used the transformations

$$\Phi(z) = \int_0^\infty \mathrm{e}^{-pz} V(p)\,\mathrm{d}p,$$

$$\Delta^2\Phi(z) = \int_0^\infty \mathrm{e}^{-pz} 2[\cosh(p) - 1]V(p)\,\mathrm{d}p.$$

The r.h.s. of (18) is a convolution defined by

$$V(p) * V(p) * V(p) = \int_0^p \int_0^{p-\tau'} V(p - \tau' - \tau)V(\tau)V(\tau')\,\mathrm{d}\tau\,\mathrm{d}\tau'.$$

The Borel transformation of the asymptotic solution (16) leads to

$$V(p) = \sum_{l=0}^{\infty} \frac{c_l}{(2l)!} p^{2l}. \tag{19}$$

Because of the convergence of coefficients $\frac{c_l}{(2l)!}$ with $l \to \infty$, divergent series $\Phi_{00}(z)$ will be made Borel summable. In terms of $V(p)$ in (19), solutions for stable and unstable manifolds are given by

$$\Phi_s(z) = \int_0^{\infty} \mathrm{e}^{-pz} V(p)\,\mathrm{d}p \tag{20a}$$

$$\Phi_u(z) = \int_0^{-\infty} \mathrm{e}^{-pz} V(p)\,\mathrm{d}p. \tag{20b}$$

Thanks to the convergence of $V(p)$, the integrals in (20a) and (20b) are convergent in the right ($\mathrm{Re}\, z > 0$) and left ($\mathrm{Re}\, z < 0$) semicircles, respectively. Hence both of them are Borel summable.

As recognized in (18), $V(p)$ has singularities at $p = \pm 2\pi \mathrm{i} n (n = 1, 2, \ldots)$. In $\mathrm{Im}\, z < 0 (|z| \to \infty)$ region, $\Phi(z)$ can be obtained by taking the p-integration along the counter-clockwise path surrounding the positive imaginary p axis and its resultant expression is given by

$$\Phi(z) = \Phi_0(z) + \Phi_1(z)\mathrm{e}^{-2\pi \mathrm{i} z} + \Phi_2(z)\mathrm{e}^{-4\pi \mathrm{i} z} + \cdots. \tag{21}$$

The expansion (21) captures exponentially small terms beyond all orders. This point will be made more explicit in terms of the difference function defined in $\mathrm{Im}\, z < 0$ as,

$$\Phi_-(z) = \Phi_s(z) - \Phi_u(z) = \int_\gamma \mathrm{e}^{-pz} V(p)\,\mathrm{d}p \tag{22}$$

where the integration path γ is indicated in figure 3(a). The poles of $V(p)$ contribute to the integration in (22), leading to the converged values of $\Phi_-(z)$ in the z region indicated in figure 3(b). Φ_- is a direct manifestation of the separatrix splitting. In the limit $|z| \to \infty$ with $\mathrm{Im}\, z < 0$ and $\mathrm{Re}\, z > 0$, Φ_u is expressed as in (21) since, in this region, $\Phi_s \sim \Phi_{00}$.

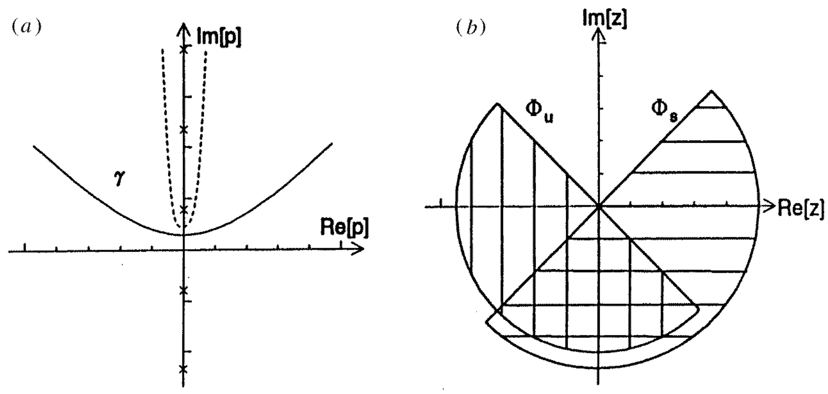

Figure 3. (a) Integration path γ in the p plane for obtaining Φ_-. γ is deformable wherever no pole distributes. (b) Regions of convergence in the z plane; vertical-, horizontal- and cross-hatched regions for Φ_u, Φ_s and Φ_-, respectively.

The Stokes phenomenon occurs on the Stokes line at $\arg z = -\pi/2$ where the asymptotic solution (21) for $|z| = \infty$ demonstrates an abrupt change. On the Stokes line, noting the relation $\Phi_u(z) = -\Phi_s(-z)$ and $\Phi_s(z^*) = \Phi_s^*(z)$, we find

$$\Phi_u(z) = -\Phi_s^*(z). \tag{23}$$

From (22) and (23), the equality

$$\Phi_-(z) = -2\,\mathrm{Re}[\Phi_u(z)] \tag{24}$$

is obtained, which implies that the real part of Φ_u leads to the separatrix splitting.

We shall proceed to substitute the expansion (21) into the internal equation in (15), deriving the equations successively in each power of the exponential:

$$\Delta^2\Phi_0(z) = 4\Phi_0^3(z) \tag{25a}$$

$$\Delta^2\Phi_1(z) = 12\Phi_0^2(z)\Phi_1(z) \tag{25b}$$

$$\ldots.$$

The asymptotic ($z \to \infty$) solution to the lowest order is given by

$$\Phi_0 = -\frac{\sqrt{2}}{2}\frac{1}{z} + \frac{\sqrt{2}}{6}\frac{1}{z^3} + \cdots. \tag{26}$$

Using $\Phi_0 \sim -\frac{\sqrt{2}}{2}\frac{1}{z}$ in (25*b*), the leading term of $\Phi_1(z)$ turns out to be $\sim z^3$. This issue reflects that $V(p)$ has a singularity described as

$$V(p) \sim \frac{k}{(p - 2\pi\mathrm{i})^4}. \tag{27}$$

In fact, we observe

$$\begin{aligned}
\Phi_- &= \int_\gamma \mathrm{e}^{-pz} V(p)\,\mathrm{d}p \\
&\sim 2\pi\mathrm{i}\,\mathrm{Re}\,s\left[\mathrm{e}^{-pz}\frac{k}{(p-2\pi\mathrm{i})^4}\right]_{p=2\pi\mathrm{i}} \\
&= \lim_{p\to 2\pi\mathrm{i}} 2\pi\mathrm{i}k\frac{1}{(4-1)!}\frac{\mathrm{d}^3}{\mathrm{d}p^3}\left\{(p-2\pi\mathrm{i})^4\frac{\mathrm{e}^{-pz}}{(p-2\pi\mathrm{i})^4}\right\} \\
&= -2\pi k\frac{1}{3!}z^3\mathrm{e}^{-2\pi\mathrm{i}z} \sim cz^3\mathrm{e}^{-2\pi\mathrm{i}z}.
\end{aligned} \tag{28}$$

$c(= -2\pi k/3!)$ is a *Stokes constant* to be evaluated. Except for this numerical factor we have succeeded in demonstrating the exponentially small term beyond all orders. By defining

$$K = \lim_{p\to 2\pi\mathrm{i}} (p - 2\pi\mathrm{i})B[z^{-3}\Phi_0(z)](p) \tag{29}$$

one may put $c = 2\pi\mathrm{i}K$ as recognized in (28). (In (29), $B[\cdot](p)$ implies Borel transformation.) The next section will be devoted to a computation of K.

4. Stokes constant

This section is concerned with a technical detail of calculating the Stokes constant (29). Therefore readers may glance over the following description and move on to section 5.

To begin with, let us define

$$A(p) = B[z^{-3}\Phi_0(z)](p). \tag{30}$$

Applying an elementary formula for the Laplace transformation, we find

$$A(p) = D_p^{(-3)} V(p) \tag{31}$$

where $D_p^{(-3)}$ implies triple integrations over variable p. Using in (31) the expansion (19) or its refined version,

$$V(p) = \sum_{k=0}^{\infty} v_k p^{2k} \qquad \text{with } v_k = \frac{c_k}{(2k)!} \tag{32}$$

one obtains

$$A(p) = \sum_{k=0}^{\infty} v_k \frac{(2k)!}{(2k+3)!} p^{2k+3} \equiv p^3 C(p) \tag{33}$$

and the resultant expansion of $C(p)$ becomes

$$C(p) = \sum_{k=0}^{\infty} b_k p^{2k} \tag{34a}$$

with

$$b_k = \frac{v_k (2k)!}{(2k+3)!}. \tag{34b}$$

Recalling the description above (21), $C(p)$ and $A(p)$ have the common singularities at $p = \pm 2\pi\mathrm{i}$. From (27) and (31), these singularities take the form $\sim (p \mp 2\pi\mathrm{i})^{-1}$. Therefore the following relation holds

$$C(p) = \frac{\chi}{(p/2\pi)^2 + 1} = \chi(1 - (p/2\pi)^2 + (p/2\pi)^4 + \cdots). \tag{35}$$

Comparing the coefficients in (34*a*) and (35), the equality

$$\chi = \lim_{k\to\infty} (-1)^k (2\pi)^{2k} b_k \tag{36}$$

is available. Noting that

$$\lim_{p\to 2\pi\mathrm{i}} ((p/2\pi) - \mathrm{i}) C(p) = \chi \lim_{p\to 2\pi\mathrm{i}} \frac{(p/2\pi) - \mathrm{i}}{(p/2\pi)^2 + 1} = -\mathrm{i}\frac{\chi}{2}$$

K is related to χ via

$$K = \lim_{p\to 2\pi\mathrm{i}} (p - 2\pi\mathrm{i}) A(p) = \lim_{p\to 2\pi\mathrm{i}} (p - 2\pi\mathrm{i}) p^3 C(p) = -\frac{(2\pi)^4}{2}\chi. \tag{37}$$

Using (34*b*) and (36) in (37), K in (37) turns out expressible in terms of the limiting value of v_k as

$$K = -\frac{(2\pi)^4}{2} \lim_{k\to\infty} (-1)^k \frac{v_k (2\pi)^{2k}}{(2k+3)(2k+2)(2k+1)}. \tag{37'}$$

The remaining business is to derive the equation for $\{v_k\}$ and to solve it numerically. Exploiting the formula

$$p^\alpha * p^\beta * p^\gamma = \frac{\alpha!\beta!\gamma!}{(\alpha+\beta+\gamma+2)!} p^{\alpha+\beta+\gamma+2}$$

with positive α, β, γ, we have

$$V(p) * V(p) * V(p) = \sum_{n=0}^{\infty} \frac{p^{2n+2}}{(2n+2)!} \sum_{j+k+l=n} v_j v_k v_l (2j)!(2k)!(2l)!. \tag{38}$$

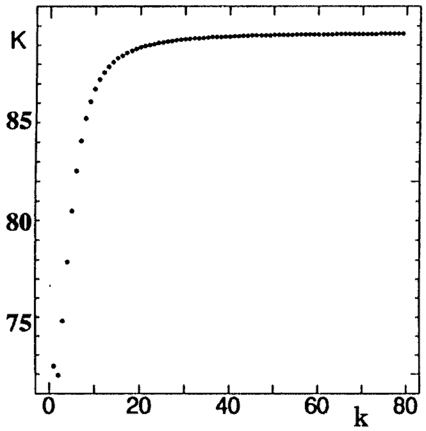

Figure 4. Convergence of Stokes constant K.

Substituting (38) into the convolution equation (18), we get a recursion equation:

$$2\left(\frac{1}{2!}-\frac{2(2m)!}{(2m+2)!}3v_0^2\right)v_m = -2\sum_{\substack{j+k=m\\0<j,k<m}}\frac{v_k}{(2j+2)!}$$
$$+\frac{4}{(2m+2)!}\sum_{\substack{j+k+l=m\\0\leqslant j,k,l<m}} v_j v_k v_l (2j)!(2k)!(2l)!. \qquad (39)$$

By numerical iteration of (39) under the initial condition $v_0 = -\sqrt{2}/2$ (which is provided by (12a) and (32)), v_m can be evaluated for large m, which will determine the converged value K in (37′). Our computation has derived the value $K \sim 89.6$ (see figure 4).

5. Matching of solutions

We are now faced with the procedure of matching the internal solution to the external one. In this context, we shall envisage the parity of the solution and the way of the Stokes constant to show up in t-plane.

Following the expansion of the internal solution in (21), the refined external solution is expected to be given by

$$\hat{y}(t,\sigma) = \hat{y}_0(t,\sigma) + S(t)\{\hat{y}_1(t,\sigma)\mathrm{e}^{-2\pi\mathrm{i}t/\sigma} + \hat{y}_2(t,\sigma)\mathrm{e}^{-4\pi\mathrm{i}t/\sigma} + \cdots\} \qquad (40)$$

where $S(t)$ describes an abrupt change of $\hat{y}(t,\sigma)$ in crossing the Stokes line at $\mathrm{Re}\, t = 0$ and is represented by the step function

$$S(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2} & t = 0\,. \\ 1 & t > 0 \end{cases} \qquad (41)$$

The substitution of (40) into the original difference equation in (5) yields

$$\Delta^2\hat{y}_0(t,\sigma) = -4\sigma^2\hat{y}_0(t,\sigma)(\hat{y}_0(t,\sigma)+a)(\hat{y}_0(t,\sigma)-a) \qquad (42a)$$
$$\Delta^2\hat{y}_1(t,\sigma) = -4\sigma^2[3(\hat{y}_0(t,\sigma))^2 - a^2]\hat{y}_1(t,\sigma). \qquad (42b)$$

In order to obtain the lowest-order solution $y_{10}(t)$ for $\hat{y}_1(t,\sigma)$, it is enough to take the lowest-order solution $y_{00}(t) = \sqrt{2}a/\cosh(2at)$ for $\hat{y}_0(t,\sigma)(= y_{00}(t) + \sigma^2 y_{01}(t) + \cdots)$ (see (9) and (11)). In this approximation, $y_{10}(t)$ satisfies a differential version of (42*b*):

$$\mathrm{d}^2 y_{10}(t)/\mathrm{d}t^2 = -4[3(y_{00}(t))^2 - a^2]y_{10}(t). \tag{42b'}$$

As discussed in section 2, (42b') has a general solution

$$y_{10}(t) = c_1 v_1(t) + c_2 v_2(t) \tag{43}$$

where $v_1(t)$, $v_2(t)$ are already given in (10). Integration constants c_1, c_2 will be determined by means of matching with Φ_{10} in the neighbourhood of $t = t_c = \mathrm{i}\pi/(4a)$. We shall embark upon this procedure.

Let us introduce a small parameter $\delta = t - t_c$. Noting the identities $\sinh[2at] = \mathrm{i}\cosh[2a\delta]$ and $\cosh[2at] = \mathrm{i}\sinh[2a\delta]$, $v_1(t)$ and $v_2(t)$ are rewritten as

$$v_1 = 2\sqrt{2}a^2 \frac{\mathrm{i}\cosh[2a\delta]}{\sinh^2[2a\delta]} \tag{44a}$$

$$v_2 = \frac{-\mathrm{i}}{8\sqrt{2}a^3}\left\{-\frac{6a\delta\cosh[2a\delta]}{\sinh^2[2a\delta]} - \frac{3\mathrm{i}\pi}{2}\frac{\cosh[2a\delta]}{\sinh^2[2a\delta]} + \frac{3}{\sinh[2a\delta]} + \sinh[2a\delta]\right\}. \tag{44b}$$

As easily observed, $v_1(t)$ is an even function of δ, while $v_2(t)$ consists of both even and odd terms. Recalling the odd parity nature of Φ_{10} (see (28)), $y_{10}(t)$ in (43) should also be an odd function of δ, which is possible so long as

$$c_1 = -\frac{3\pi}{64a^5}\,\mathrm{i}c_2 \tag{45}$$

is satisfied. c_2 itself is related to the Stokes constant by matching y_{10} with Φ_{10} on the negative imaginary axis of the z plane (i.e. on Stokes line) as

$$\frac{1}{2}\frac{\sigma}{\mathrm{i}} y_{10}\mathrm{e}^{-2\pi\mathrm{i}t/\sigma} \sim \mathrm{Re}[\Phi_u(z)]. \tag{46}$$

Thanks to (44),

$$y_{10} = -\frac{\mathrm{i}\sqrt{2}}{5}c_2\delta^3(1 + O(\delta^2)) = -\frac{\mathrm{i}\sqrt{2}}{5}c_2\sigma^3 z^3(1 + O(\delta^2)).$$

Using in (46) this fact together with (24) and (28), we have

$$-\frac{\sqrt{2}}{5}c_2\sigma^4 z^3 \mathrm{e}^{\pi^2/2\sigma a}\,\mathrm{e}^{-2\pi\mathrm{i}z} \sim -cz^3\,\mathrm{e}^{-2\pi\mathrm{i}z} \tag{46$'$}$$

and hence

$$c_2 = \frac{5\sqrt{2}}{2}\frac{c}{\sigma^4}\mathrm{e}^{-\pi^2/2\sigma a}. \tag{47}$$

Since we already know the value of Stokes constant $c = 2\pi\mathrm{i}K$ (see the final issue of the previous section), the value of c_2 in (47) is also determined.

There is an additional contribution arising from another singularity closest to real t axis at $t = t_c^* = -\frac{\mathrm{i}\pi}{4a}$ (see above (12)). This contribution is merely a complex conjugate of the existing result for $y_{10}\mathrm{e}^{-2\pi\mathrm{i}t/\sigma}$. Combining a pair of contributions, the asymptotic behaviour of y_u on the real t axis is eventually given by

$$y_u = \sum_{n'=0}^{\infty}\sigma^{2n'}y_{0n'} + 2S(t)\,\mathrm{Re}\left[\sum_{n=1}^{\infty}\sum_{n'=0}^{\infty}\sigma^{2n'}y_{nn'}\mathrm{e}^{-2n\mathrm{i}\pi t/\sigma}\right]. \tag{48}$$

Therefore y_u is explicitly written up to terms $l(\equiv n + n') = 1$ as

$$y_u = y_{00} + \sigma^2 y_{01} + 2S(t)\{c_1 v_1(t)\cos(2\pi t/\sigma) - \mathrm{i}c_2 v_2(t)\sin(2\pi t/\sigma)\} \tag{49}$$

with $v_1(t)$ and $v_2(t)$ in equations (10).

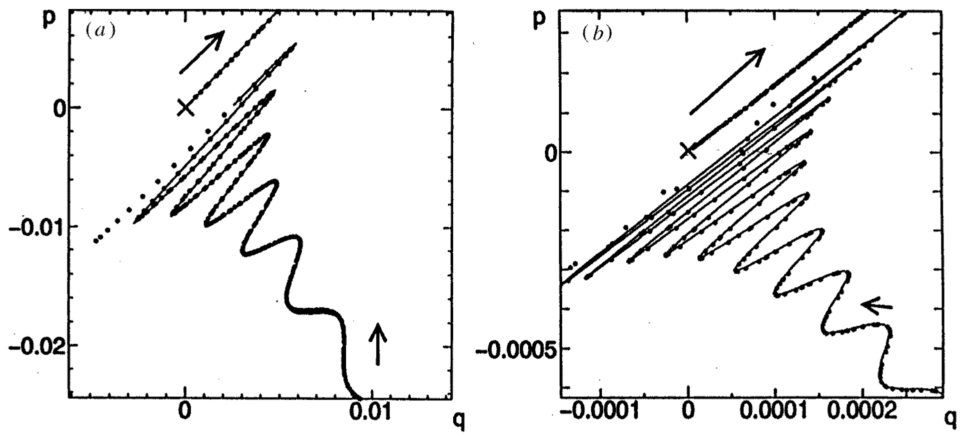

Figure 5. Unstable manifold with a hyperbolic fixed point (×). Arrow indicates direction of journey. Full curve for the analytical results and a series of dots for the results of numerical iteration of the map: (*a*) $a = 1$, $\sigma = 0.2$; (*b*) $a = 1$, $\sigma = 0.15$.

6. Homoclinic structures and intersection angle

Coming back to the symplectic map in (3), the solution for (q_u, p_u) is constructed by the replacement

$$(q_u, p_u) = (y_u(t), (y_u(t) - y_u(t-\sigma))/\sigma). \tag{50}$$

Figure 5 shows a nice agreement between the asymptotic analytical result and the issue of the numerical iteration of the map (3). The sequence of dots are obtained by numerical iteration of the map (3) for an assembly of initial points on the linearized unstable manifold at HFP (: on one of the eigenvectors of monodromy matrix (4)). As for the analytical result, it should be noted: if $\sigma \ll 1$, the complete Stokes phenomenon occurs ensuring an abrupt change of S in (41) in crossing the Stokes line at $\mathrm{Re}\,t = 0$. For larger values ($\sigma \sim 0.1$), we shall see more or less incomplete Stokes phenomenon, i.e. a mild growth of $S(t)$ in a narrow region around $\mathrm{Re}\,t = 0$. However, this problem can be resolved by exploiting an appropriate constant value for $S(t)$ for $t > 0$ in (41). In fact, with a choice of $S = 0.82$, (49) proves to work very well for any value of σ between 0.1 and 0.3 (see figure 5). Figure 5(*a*) and (*b*) are the magnification of the vicinity of HFP. The unstable manifold starting from HFP, after executing a long clockwise journey, comes back again to the vicinity of HFP but accompanied by violent undulations. The asymptotic analytical line proves to fit the result of the numerical iteration of (3), describing the stretching of the area enclosed by the stable and unstable manifolds. When σ is decreased, violent undulations begin to occur in the further vicinity of HFP: see the extremely small scale unit $\sim 10^{-4}$ for both p and q axes in figure 5(*b*).

Noting that the stable manifold is merely the time reversal of the unstable manifold, let us proceed to consider an angle for the intersection between the stable and unstable manifolds at the first homoclinic point at $t = 0$ where the unstable (stable) manifold begins (ceases) oscillations. Let $y_u(t)$ be divided into the even- and odd-parity parts as

$$y_u(t) = I(t) + E(t) \tag{51}$$

where $I(t) = y_{00} + \sigma^2 y_{01} + \cdots$ and $E(t) = 2S(t)\{c_1 v_1(t)\cos(2\pi t/\sigma) - \mathrm{i} c_2 v_2(t)\sin(2\pi t/\sigma)\}$. Then the unstable and stable manifolds are constructed respectively by $q_u(t) = I(t) + E(t)$

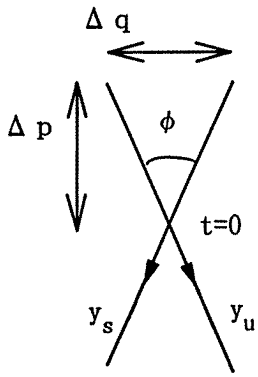

Figure 6. Intersection angle ϕ.

and $q_s(t) = I(t) - E(t)$ with $p_u(t)$ and $p_s(t)$ expressed by (50). In the neighbourhood of the first homoclinic point, we shall concentrate on the small time $0 < \delta t_1, \delta t_2 \ll \sigma \ll 1$ satisfying $p_u(\delta t_2) = p_s(\delta t_1)$ (see figure 6). Since $\delta t_1 \sim \delta t_2 \sim \delta t$, the intersection angle is given by $\phi = \Delta q/\Delta p = (q_s(\delta t) - q_u(\delta t))/(p_s(\delta t) - p_s(0))$. Noting $\dot{E}(0)$, $\dot{E}(\sigma) \ll \dot{I}(\sigma)$ ($\dot{A}$ means $\delta A/\delta t$) and using $S(0) = 1/2$ (see (41)), ϕ becomes

$$\phi = 2\sigma \dot{E}(0)/\dot{I}(\sigma) = \pi(8a^6\sigma)^{-1}|c_2| = \frac{5\sqrt{2}\pi^2 K}{8a^6\sigma^5} \exp\left(-\frac{\pi^2}{2a\sigma}\right)$$

showing an essential singularity at $\sigma = 0$.

6.1. Comparison with other maps

The refined method of asymptotic expansion beyond all orders developed above is identical to the one applied to the standard map [6] and Henon map [8], which are given by

$$x_{n+1} = x_n + \alpha y_{n+1} \qquad y_{n+1} = y_n + \beta \sin(x_n) \tag{52}$$

and

$$x_{n+1} = -y_n + \beta x_n - \lambda x_n^2 \qquad y_{n+1} = x_n \tag{53}$$

respectively. These maps, after the rescalings as $x_{n\pm1} = y(t \pm \sigma)$ with $\alpha = \beta = \sigma$ in (52) and $x_n = \sigma^2 y(t)/(1 - \sigma^4)$ with $\beta = 2(1 - \sigma^2)$, $\lambda = \sigma^4 - 1$ in (53), are rendered to

$$\Delta^2 y(t)/\sigma^2 = \sin[y(t)] \tag{52$'$}$$

and

$$\Delta^2 y(t)/\sigma^2 = y^2(t) - 2y(t) \tag{53$'$}$$

respectively. In the case of the standard map, the intersection angle was obtained [6]. To our knowledge, however, the heteroclinic structures have not been depicted by using the asymptotic analytical expressions. In the case of the Henon map, the homoclinic structure is shown [8]. Since the intersection angle in this case has not yet been provided, however, we have computed it below. For the three maps (i.e. the present, standard and Henon ones), we find the intersection angle takes the universal form

$$\phi = (a_1/\sigma^\mu) \exp[-a_2/\sigma]. \tag{54}$$

In (54), a_1 and a_2 are positive real constants and μ is a positive integer. Their values are $(\mu, \alpha_1, \alpha_2) = (5, \sqrt{5}\pi^2 K/(8a^6), \pi^2/(2a))$, $(3, 24\pi^4 K, \pi^2)$ [6] and $(8, 56\pi^2 K/3, \sqrt{2}\pi^2)$, respectively, for the present, standard and Henon maps. (The K value itself depends on each map, taking $K \sim 89.6$, 1.503 [6] and 7374 [8] for the present, standard and Henon maps, respectively.)

7. Conclusions

The time-discrete classical dynamics or the symplectic map for the system with a double-well potential has been analysed, by resorting to the refined *asymptotic expansion beyond all orders* based on theoretical tools of Borel summability and Stokes phenomenon. The homoclinic structures are shown to be nicely described by the analytical expression in (48) and (49) including only $l = 0, 1$ terms. In particular, the stretching-type oscillations appearing in the extreme vicinity of the hyperbolic fixed point recovers excellently the issue of numerical iteration of the map (3) (note the scale unit $\sim 10^{-4}$ in figure 5(*b*)). Inclusion of higher-order ($l \geqslant 2$) terms is anticipated to derive the folding mechanism and thereby to complete Birkhoff–Smale's horse-shoe mechanism for genesis of chaos in conservative dynamical systems. By comparative study on the present, standard and Henon maps, the intersection angle of the first homoclinic or heteroclinic points is shown to take a universal form showing an essential singularity at $\sigma = 0$. The latest articles [9, 10] which treat symplectic discretizations of second-order variational equations, indicate the upper bound for the exponential splitting. The concrete and explicit values for the intersection angle in the present work satisfy the existing inequality.

The present framework based on the updated method of *asymptotic expansions beyond all orders*, bearing neither peculiarity nor particular difficulty of the model, will be very useful for the analytical study of chaos in general, which has been investigated mostly by numerical computations or scaling arguments for a long time. In this context one can also apply the present method to dissipative systems, since symplectic properties have not been used in the present work. It is further desirable to proceed to study analytically the quantum and semiclassical analogues [11, 12] of the homoclinic structures.

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References

[1] Poincaré H 1890 *Acta Math.* **13** 1
[2] Lazutkin V F, Schachmannski I G and Tabanov M B 1988 *Physica* **40D** 235
[3] Gelfreich V G, Lazutkin V F and Svanidze N V 1994 *Physica* **71D** 82
[4] Amit C, Ching E S C, Kadanoff L P and Rom-Kedar V 1992 *J. Nonlinear Sci.* **2** 9
[5] Kruskal M D and Segur H 1991 *Stud. Appl. Math.* **85** 129
[6] Hakkim V and Mallick K 1993 *Nonlinearity* **6** 57
[7] Tovbis A 1994 *Commun. Math. Phys.* **163** 245
[8] Tovbis A, Tsuchiya M and Jaffe C 1996 *Preprint*
[9] Angenent S 1993 *Symplectic Geometry (London Mathematical Society Lecture Notes 192)* ed D Salamon, p 5
[10] Fiedler B and Scheurle J 1996 *Memoirs of the AMS* p 570
[11] Nakamura K 1993 *Quantum Chaos—A New Paradigm of Nonlinear Dynamics* (Cambridge: Cambridge University Press)
[12] Gutzwiller M C 1990 *Chaos in Classical and Quantum Mechanics* (Berlin: Springer)